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EDITED BY

W. H. METZLER

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EUGENE R. SMITH JONATHAN T. RORER

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EQUATIONS.*

By JAMES M. TAYLOR.

THE AXIOMS OF EQUALS NOT APPLICABLE TO EQUATIONS.

The statement that two expression denote the same number is called an equality. This name is precise and descriptive; equally so are identity and equation as names respectively of an unconditional equality and a conditional equality. The beginner in algebra should clearly distinguish between these two kinds of equalities.

Some of the characteristic differences between equations and identities are the following:

The *numbers* in an *equation* are classified as *knows* and *unknows*.	The numbers in an identity are not classified as knows and unknowns.
An equation is a conditional equality, *i. e.*, it holds true only for a limited number of values of its unknown letters.	An identity is an unconditional equality, *i. e.*, it holds true for all values of its letters, or all values between certain limits.
An equation is to be solved, *i. e.*, we are to find the values of its unknown or unknowns.	An identity is to be proved, *i. e.*, we are to prove that its members are identical expressions.
In solving equations we use	In proving identities we use

the principles of equivalency.	the axioms of equals, or the principles of identical expressions.
Equations are chiefly useful in stating and solving problems.	Identities are chiefly useful in transforming and solving equations.

The pupil has proved identities and used them in his first lessons in arithmetic. Counting two implies the identity $1+1=2$. In proving identities we use the axioms of equals, or preferably the axiomatic principles of identical expressions. The latter emphasize what must be maintained, identical expressions in the two members.

In solving an equation we seek the values, or sets of values, of its unknowns which will render the equation an identity. Some equations are so simple that we discover their solutions by inspection. But, in general, to solve an equation we must derive from it simpler equations. That any derived equation may be of any use, we must know that it is equivalent to the given equation, or at least that its solutions include all those of the given equation. By the axioms of equals we can not determine whether any derived equation fulfills this condition or not. To illustrate this fact, let us consider some equations derived by applying these axioms to equations.

Suppose we have given the equation

$$1-\frac{x^2}{x-1}=\frac{1}{1-x}-6. \qquad (1)$$

Multiplying the conditionally equal members of (1) by $x-1$. and transposing we obtain

$$x^2-7x+6=0. \qquad (2)$$
$$\therefore x=1 \text{ or } 6.$$

If the axioms proved equivalency, we would conclude that equations (1) and (2) were equivalent. But the derived equation (2) is not equivalent to (1); for $x=1$ does not satisfy (1). But $x=6$ does satisfy (1); hence the solutions of the derived equation (2) include one solution of (1). But as one solution was introduced, the question arises whether any solution was *lost* in the operation. To answer this question and to

explain how the new solution was introduced, we need the doctrine of equivalency.

As another illustration take the two equations,

$$x - 1 = 2, \qquad (3)$$

and

$$x + 1 = 5. \qquad (4)$$

Multiplying the members of (3) by the corresponding members of (4) we obtain

$$x^2 - 1 = 10, \quad \text{or} \quad x = \pm\sqrt{11} \qquad (5)$$

The solution of equation (3) is 3 and that of (4) is 4, while the solutions of the derived equation (5) are $\pm\sqrt{11}$. Hence by multiplying the conditionally equal members of (3) by those of (4) we lost both the solution of (3) and that of (4) and introduced the two solutions of (5).

Dividing the members of (3) by those of (4), we obtain

$$\frac{x-1}{x+1} = \frac{2}{5}, \quad \text{or} \quad x = \frac{7}{3}. \qquad (6)$$

Here the solutions of (3) and (4) were lost and the solution of (6) was introduced.

Adding the members of (3) to those of (4) we obtain

$$2x = 7, \quad \text{or} \quad x = 7/2. \qquad (7)$$

Here the solutions of (3) and (4) were lost, and the solution of (7) was introduced.

Subtracting the members of (3) from those of (4) we obtain

$$0x + 2 = 3, \quad \text{or} \quad 0x = 1, \qquad (8)$$

which has no solution, or is impossible.

In this case the solutions of (3) and (4) were lost and no solution was gained.

Again from (3) and (4) we obtain,

$$x + 1 = 4, \qquad (9)$$

and

$$x - 1 = 3. \qquad (10)$$

Multiplying the members of (9) by those of (10) we obtain

$$x^2 - 1 = 12, \quad \text{or} \quad x = \pm\sqrt{13}. \qquad (11)$$

Here the two solutions of (9) and (10) were lost and the two solutions of (11) were introduced.

Dividing the members of (9) by those of (10) we obtain

$$\frac{x+1}{x-1}=\frac{4}{3}, \quad \text{or} \quad x=7. \qquad (12)$$

In this case the two solutions of (9) and (10) were lost, and the solution of (12) was introduced.

Adding the members of (9) to those of (10) we obtain

$$2x=7, \quad \text{or} \quad x=7/2. \qquad (13)$$

Here the two solutions of (9) and (10) were lost, and the solution $7/2$ was introduced.

By subtracting the members of (9) from those of (10) we obtain the impossible equality, $0x-2=-1$, or $0x=1$. Writing the equations in the form

$$x-3=0, \qquad (14)$$

and

$$x-4=0, \qquad (15)$$

and then multiplying as above we obtain

$$(x-3)(x-4)=0. \qquad (16)$$

The derived equation (16) is equivalent to the two equations (14) and (15) jointly; hence no solution was lost or gained. But if we change the order of the members of (14) and then multiply we obtain the identity $0=0$.

Adding the members of (14) to those of (15) we obtain

$$x=7/2.$$

Subtracting the members of (14) from those of (15) we obtain the impossible equation $0x=1$.

Of the operations above, the doctrine of equivalency of equations would allow none except the first multiplication with equations (14) and (15), by which no solution was lost or gained.

It might be objected that the x's in the equations above denoted different numbers. But let us take two equations which are equivalent; *e. g.*,

$$x - 1 = 2, \qquad (17)$$

and

$$x - 1 = 2. \qquad (18)$$

Multiplying the members of (17) by those of (18) we obtain

$$x^2 - 2x + 1 = 4, \quad \therefore x = 3 \text{ or } -1. \qquad (19)$$

Here one solution was lost and one introduced.

Dividing the members of (17) by those of (18) we obtain the identity $1 = 1$.

Subtracting the members of (17) from those of (18) we obtain the identity $0 = 0$.

Changing the order of the members of (18) and adding we obtain the identity $x + 1 = x + 1$.

Again squaring the members of the equation

$$-2 + \sqrt{2x + 8} = 2\sqrt{x + 5}, \qquad (20)$$

and then squaring again we obtain

$$x^2 - 16 = 0, \quad \text{or} \quad x = \pm 4. \qquad (21)$$

But neither $+4$ nor -4 will render (1) an identity. Hence both solutions of (21) were introduced by squaring. The axioms give no hint of this fact. In fact until very recently, both of the solutions of (21) would have been given by text-book writers as solutions of equation (20).

The doctrine of equivalency proves that if equation (20) has any solution, it is included among the solutions of (21). Hence (20) is to be tested for each solution of (21). These cases will suffice to illustrate that by the axioms of equals, we can not prove what we want to know concerning any derived equation. The question arises, "Are the axioms of equals ever applicable to equations? For an answer we have not far to look; the answer is in the meaning of the word *equals*. In the axioms, equals, or equal numbers, mean unconditionally equal numbers. The members of an identity denote such numbers. Hence these axioms are applicable to identities. But the members of an equation do not denote unconditionally equal numbers. Hence these axioms are not applicable to equations.

To be applicable to equations the axioms must still be true when in them the phrase "conditionally equal" is substituted for the word "equal." Making this substitution in the axiom, "If each of two numbers is equal to the same number, they are equal to each other," we have.

"If each of two numbers is conditionally equal to the same number they are conditionally equal to each other."

This does not sound like an axiom, nor does it act like one. In fact it is not true.

E. g., if we take the two equations,

$$x - 1 = 3 \quad \text{and} \quad x - 5 = 3,$$

and apply this proposition as an axiom we obtain

$$x - 1 = x - 5, \quad \text{or} \quad 0x = 4,$$

which is impossible for any value of x.

Thus we see that the futile results in equalities (5), (6), (7), (8) are obtained from (3) and (4) not by what the axioms of equals really say, but by what they are slanderously charged with saying.

If in the equations (3), (4), (9), (10), (14), (15), (17), (18), the conditional element were removed by substituting for each unknown its value, then the results obtained by each operation in accord with an axiom of equals would be an identity.

Thus if any number of operations are performed on the members of an identity or of identities and each is in accord with some axiom of equals, the derived equality will be an identity.

But if any number of operations are performed on the members of an equation or of equations, each of which is admissable on identities the derived equality may or may not be an equation equivalent to the given equation or equations.

Hence from the fact that all the operations performed on an equation or equations are admissable on identities, our futile conclusion is that the derived equality *may* or *may not* be an equation equivalent to the given equation or equations.

Thus when a pupil gives an axiom of equals as a proof of equivalency he gives a false reason, which is a thousand times worse than no reason at all.

If any reason is given for the equivalency of two equations, let it be a principle of equivalency. The objection that the beginner can not understand the principles of equivalency and prove them is not well founded. The first principles of equivalency are very simple in thought, and just as easy of application as the axioms. Their general proof should not be required of the beginner, but he should often verify them in particular examples, and thus make their thought familiar and convince himself of their truth. The word equivalent or equivalency should not be used at first. Each principle should be illustrated, and then stated, for equations in one unknown, somewhat as follows:

I. If two expressions are identical and either is substituted for the other in an equation, the unknown has the same values in the derived equation as in the given one.

II. If identical expressions are added to or subtracted from both members of an equation, the unknown has the same values in the derived equation as in the given one.

III. If both members of an equation are multiplied or divided by the same known expression, not denoting zero, the unknown has the same values in the derived equation as in the given one.

This form of statement has the merit of emphasizing what the pupil's attention should be fixed upon, viz., the values of the unknown. These values are the game he is seeking and his eye should be kept upon them. Only the doctrine of equivalency can keep this game in sight.

The Direct Problem of Making Equations.

The natural, common-sense and pedagogical order is to solve and to study first the direct problem and afterwards the inverse problem. Addition comes before subtraction, multiplication before division or factoring, and addition of fractions before decomposition of fractions. We learn how to make differential equations before we try to solve them. According to this fundamental principle, the direct problem of making quadratic and higher equations should precede the inverse problem of solving them. But in equations the common practice is to ignore the direct problem and to take up first the difficult inverse problem. This neglect of the direct problem is the more to be regretted, because the simple problem of making

equations which have given roots brings clearly into view the fundamental principles and the fundamental method of solving equations and also the nature of this inverse problem. Moreover the study of this simple direct problem makes clear some of the most important properties of equations, such as, the number of its roots, the relation of its roots to its coefficients, the appearance of surd, imaginary and complex roots in conjugate pairs, etc.

Having given pairs of particular numbers the pupil should make quadratic equations which have these numbers as roots. At first the given roots should be real and commensurable, then conjugate surds, then conjugate imaginaries and then conjugate complex numbers. The pupil should carefully note that, in each of the above cases, the coefficients of each equation are real and commensurable. Then let equations be made which have such pairs of roots as are not included in the cases above, and have the pupil note the fact that in each equation the coefficients are not all real and commensurable. The pupil is thus prepared for solving and studying the general problem below:

To find the equation whose solutions are a_1 and a_2 where a_1 and a_2 are any numbers whatever.

The equation whose solution is a_1 is

$$x - a_1 = 0. \tag{22}$$

The equation whose solution is a_2 is

$$x - a_2 = 0. \tag{23}$$

By multiplying we obtain

$$(x - a_1)(x - a_2) = 0. \tag{24}$$

But we have the identity

$$(x - a_1)(x - a_2) \equiv x^2 - (a_1 + a_2)x + a_1a_2. \tag{25}$$

From equation (24) by identity (25) we obtain

$$x^2 - (a_1 + a_2)x + a_1a_2 = 0. \tag{26}$$

Equation (26) is equivalent to (24) and therefore to (22) and (23) jointly.

Some of the obvious conclusions from this problem are the following corollaries:

COR. 1. From equation (26) it follows that every equation that has two given roots is a quadratic equation.

Again from the identity (27)

$$\begin{aligned} x^2 + 2px + q &\equiv (x+p)^2 - (p^2 - q) \\ &\equiv (x + p + \sqrt{p^2 - q})(x + p - \sqrt{p^2 - q}), \quad (27) \end{aligned}$$

it follows that every quadratic equation can be put in the form of (24).

Hence conversely every quadratic equation has two roots and only two roots.

A glance at the solution above would lead the pupil to infer that the degree of any equation and the number of its roots are equal.

COR. 2. From equation (26) we see that when in a quadratic equation the coefficient of x^2 is $+1$, the coefficient of x is minus the sum of the roots, and the known term is the product of the roots.

COR. 3. In equation (26) the coefficients $-(a_1 + a_2)$ and a_1a_2 are real and commensurable when the roots a_1 and a_2 are both real and commensurable, or when they are conjugate surds, conjugate imaginaries, or conjugate complex numbers, and in no other case.

Hence conversely when the coefficients of a quadratic equation are real and commensurable, its two roots are real and commensurable, or they are conjugate surds, conjugate imaginaries, or conjugate complex numbers.

From Cor. 2 or equation (26), the equation which has any two given roots can be written out, and thus the relation between the roots and the coefficients of an equation made familiar.

To solve the general quadratic equation (26) we retrace our steps to the linear equations (22) and (23). These equations, or the values of x given in them, are the solutions of (26). We thus see that the solving of an equation involves the same principles of equivalency as its making and that where the making involves finding the product of given linear factors the solving involves finding the linear factors of a given product.

Thus the pupil sees that factoring is not one of several ways

of solving a quadratic or higher equation, but that fundamentally factoring is the *only* way of solving such equations. He can not be too strongly impressed with this fact, so that whatever devices he may later employ in solving such equations he will recognize them as devices for factoring.

The great gain in first studying the direct problem in equations is not that some properties of equations are thus more easily proved, but that these properties are thus made more evident, tangible, real and familiar to the pupil at the very beginning of his study of quadratics and that from the first he has a clear idea of his problem and the fundamental method of solving it.

Substitution and Factoring in Solving Systems.

The simplest system of n equations is one of the form of system (A),

$$\left.\begin{array}{l} x_1 = a_1 \ (1) \\ x_2 = a_2 \ (2) \\ \dots\dots\dots\dots \\ x_n = a_n \ (n) \end{array}\right\} \qquad (A)$$

in which each of its n equations contains but one unknown and is solved for that unknown. To indicate that two or more equations are simultaneous and form a system they should be joined by a brace as in (A).

To solve a system of n linear equations in n unknowns is to find a system of form (A) which is equivalent to the given system. To find such a system we may proceed as follows: Solve for x any equation of the system, say the first, and substitute this value of x in each of the other equations, then in the second system x will not appear in any equation after the first. In this second system solve any equation after the first, say the second, for y, and substitute this value for y in each of the other equations, then in the third system y will not appear in the first equation, and neither x nor y will appear in any equation after the second. Continuing this operation there will be obtained finally a system of the form of (A). All the systems obtained as above will be equivalent. Hence the last system will be the solution of the first.

The operation outlined above is called elimination by substitution. The principle of substitution upon which it is based

should be stated at the outset, but at first no proof need or should be given. All the other so-called methods of elimination are but modified forms of this fundamental method by substitution. *E. g.*, elimination by comparison is clearly a case of substitution. Elimination by addition or subtraction is simply a way of substituting for certain multiples of the unknown in one equation, the values of these multiples as obtained from the other equations of the system. Elimination by indeterminate coefficients is a modified form of elimination by addition. Elimination by division is simply a way of substituting in one equation of the system, the value of a combination of some of its unknowns as obtained from another equation of the system.

Since the principle of substitution underlies all the so-called different methods of elimination, the method of substitution should be taught first; for only thus will the pupil gain at the outset, a clear idea of the fundamental principle underlying all the methods, and see clearly the relation of the other methods to this fundamental one.

But the common practice in our teaching is to present first of all the so-called method of elimination by addition. This blinds the student to the one principle underlying all his work in elimination and starts him on his career of working without thinking in the solution of systems.

If a system contains one or more equations above the first degree, and each of them can be written in the type-form, " The product of two or more linear factors equal to zero," it is easy to write the linear systems which jointly are equivalent to the given system. If a system contains one or more equations which are above the first degree, and these equations can not be factored, then the first problem is to find by sustitution a new system equivalent to the given system, whose equations above the first degree can be factored. This being done we can write and solve the linear systems which jointly are equivalent to the given system. Thus we see that factoring is not only the fundamental method of solving quadratic and higher equations in one unknown, but it is the fundamental method of solving systems which involve equations above the first degree.

In the study of equations and systems therefore we would emphasize the following points:

1. Do not misuse the axioms of equals by applying them to equations.

2. Make clear and familiar the fundamental principles of equivalency of equations and systems.

3. Discover the fundamental properties of quadratic and higher equations by solving and studying the direct problem of making such equations.

4. Emphasize that to solve an equation or a system above the first degree we must first find the linear equations or linear systems which jointly are equivalent to the given equation or system, and that fundamentally the only way of deriving these linear equations or systems is by factoring.

5. In solving systems use first the method of elimination by substitution and as each of the other methods is employed show that it is but a convenient mode of effecting substitutions.

Thus we would give our pupils a better chance of becoming thinking workers in mathematics. Our problem in teaching mathematics to-day is not so much to make the courses easy for the unthinking, as it is to make clear the fundamental principles of the science, and to stimulate our pupils to grasp, to prove and to use these principles.

Colgate University,
Hamilton, N. Y.